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Supersymmetric Deformations of G_2 Manifolds from Higher-Order Corrections to String and M-Theory

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Abstract: The equations of 10 or 11 dimensional supergravity admit supersymmetric compactifications on 7-manifolds of G_2 holonomy, but these supergravity vacua are deformed away from special holonomy by the higher-order corrections of string or M-theory. We find simple expressions for the first-order corrections to the Einstein and Killing spinor equations in terms of the calibrating 3-form of the leading-order G_2-holonomy background. We thus obtain, and solve explicitly, systems of first-order equations describing the corrected metrics for most of the known classes of cohomogeneity-one 7-metrics with G_2 structures.

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Contents

1. Introduction

The low-energy limits of string/M-theory are supergravity theories in ten or eleven dimensions. These supergravity theories admit solutions of the form $(\text{Minkowski})_d \times K_n$, where K_n is a Ricci-flat space of dimension n, and $d + n$ is equal to the total spacetime dimension. When K_n is compact, a solution of this type can be interpreted as a Kaluza-Klein vacuum, which is supersymmetric if K_n admits covariantly-constant spinors. Put another way, supersymmetric Kaluza-Klein vacua arise if K_n is a space having an appropriate *special holonomy*. The most studied case in string theory is when $n = 6$, and K_6 is a Ricci-flat Kähler, *i.e.* Calabi-Yau (CY), manifold. The most natural analogue in M-theory arises for $n = 7$, in which case one is interested in 7-manifolds K_7 with the exceptional holonomy G_2. Of course, one can also consider G_2 compactifications in the context of string theory, and we shall take this point of view initially, returning to M-theory in the final Section.

Supergravity is just the leading term in an effective action obtained, in the case of string theory, by integrating out the massive-modes of the string. The effective action

is thus an expansion in derivatives or, equivalently, in powers of the inverse string tension α'. If we focus on the part of the effective action relevant to graviton scattering amplitudes then the nth term in its expansion has the form $(\alpha')^{n-1}R^n$ where R stands for the Riemann tensor. This term has no effect on tree-level amplitudes with less than n external particles, so we would need to consider at least the n-point amplitudes to determine it; alternatively, it can be determined by an n-loop computation of the beta-function for the associated (1,1)-supersymmetric sigma-model. At the tree-level in string perturbation theory the massive modes of the string contribute only to amplitudes with at least four external gravitons, so R^2 and R^3 corrections may be absent. They *are* absent for the type II superstring theories, in agreement with the three-loop finiteness of (1,1) supersymmetric sigma models with a Ricci-flat target space so, in these cases, the first non-trivial correction is an $\alpha'^3 R^4$ term. In the string conformal frame, this term must appear with a factor of $e^{-2\phi}$, where ϕ is the dilaton, so the graviton/dilaton part of the tree-level effective Lagrangian is

$$\mathcal{L} = \sqrt{-g}\, e^{-2\phi} \left(R + 4(\partial\phi)^2 - c\,\alpha'^3\, Y \right) \tag{1.1}$$

for a known constant c, proportional to $\zeta(3)$, and a known scalar Y that is quartic in the Riemann tensor of the 10-dimensional spacetime. The corresponding equations of motion may be written

$$\begin{aligned} R_{\mu\nu} + 2\nabla_\mu \nabla_\nu\, \phi &= c\,\alpha'^3\, X_{\mu\nu} \\ \Box\, \phi - 2\,(\partial\phi)^2 &= \frac{1}{2} c\alpha'^3\, \left(Y - g^{\mu\nu} X_{\mu\nu}\right) \end{aligned} \tag{1.2}$$

where D_μ is the standard covariant derivative, and

$$\sqrt{-g}\, X_{\mu\nu} = e^{2\phi} \frac{\delta}{\delta g^{\mu\nu}} \int d^{10}x \sqrt{-g}\, e^{-2\phi}\, Y. \tag{1.3}$$

The explicit form of Y was first found from a four-loop sigma-model computation [1, 2], and this result was partially confirmed by a computation of the 4-point graviton scattering amplitude in string theory [3, 4]. There is a considerable measure of ambiguity in the choice of expression for Y, since one is free to perform field redefinitions at order α'^3 that modify Y by terms that vanish upon use of the leading-order equations of motion.

For our purposes, it is not necessary to know the fully 10-dimensional form of Y. As we are principally concerned with the effect of Y on supergravity solutions of the form $(\text{Minkowski})_3 \times K_7$, it would be sufficient to restrict the Riemann tensor to be non-zero on K_7. However, it is convenient to consider G_2 compactifications as special cases of

solutions of the form $(\mathrm{Minkowski})_2 \times M_8$, where M_8 is an 8-manifold that happens to be a product of the form $\mathbb{R} \times K_7$, and to allow the Riemann tensor to be non-zero on M_8. This has the advantage that it includes the possibility that $M_8 = K_8$, where K_8 is an 8-manifold of Spin(7) holonomy, although we will have little to say about this case in the present paper. With this understood, let $R_{ijk\ell}$ be the components of the Riemann tensor on M_8 and define t_8 to be the tensor such that

$$t_8^{i_1 j_1 i_2 j_2 i_3 j_3} A_{i_1 j_1} A_{i_2 j_2} A_{i_3 j_3} A_{i_4 j_4} = 24 \left[\mathrm{tr} A^4 - \tfrac{1}{4} \left(\mathrm{tr} A^2 \right)^2 \right] \tag{1.4}$$

for any antisymmetric 8×8 matrix A with entries A_{ij}. Then field variables can be chosen so that Y is given by

$$Y \propto \left[t_8^{i_1 \ldots i_8} t_8^{j_1 \ldots j_8} - \tfrac{1}{4} \varepsilon^{i_1 \ldots i_8} \varepsilon^{j_1 \ldots j_8} \right] R_{i_1 i_2 j_1 j_2}\ R_{i_3 i_4 j_3 j_4}\ R_{i_5 i_6 j_5 j_6}\ R_{i_7 i_8 j_7 j_8}\,. \tag{1.5}$$

As we shall see later, the choice of variables implied by (1.5) has the virtue that underlying structures associated with special holonomy backgrounds are preserved, even though the special holonomy itself is modified or lost under the effect of the α'^3 corrections.

It was pointed out in [3] that (1.5) can be written as a Berezin integral over the 16 components of a Grassmann-odd Majorana SO(8) spinor ψ. Let $\tilde{\Gamma}^i$ $(i = 1, \ldots, 8)$ be SO(8) Dirac matrices, and $R_{ijk\ell}$ the components of the Riemann tensor of the metric on M_8. Then

$$Y \propto \int d^{16}\psi \, \exp \left[\tfrac{1}{2} \left(\bar{\psi}\, \tilde{\Gamma}^{ij}\, \psi \right) \left(\bar{\psi}\, \tilde{\Gamma}^{k\ell}\, \psi \right) R_{ijk\ell} \right]. \tag{1.6}$$

where $\bar{\psi} = \psi^\dagger$. In a basis for which the Dirac matrices $\tilde{\Gamma}^i$ are real, the spinor ψ is real and $\bar{\psi} = \psi^T$.

Before proceeding with our analysis of the corrections to special-holonomy backgrounds, it is of interest to see what can be established in general. Let us assume that the dilaton is constant to leading order, so that

$$\phi = \phi_0 + \alpha'^3\, \phi_1. \tag{1.7}$$

The functional variation (1.3) then yields

$$X_{ij} = \tilde{X}_{ij} + \nabla^k \nabla^\ell X_{ikj\ell} \tag{1.8}$$

where $X_{ikj\ell}$ is a tensor cubic in the curvature, antisymmetric within first and second index pairs and symmetric under pair interchange; $\tilde{X}_{ij}$ is a symmetric tensor quartic in curvatures which is traceless in eight dimensions (as one can see from the vanishing Weyl weight of $\sqrt{-g} Y$). The tensor $\tilde{X}_{ij}$ arises from the variations of "explicit" metrics

in $\sqrt{-g}Y$ (*i.e.* metrics in $\sqrt{-g}$ or those used to contract curvature indices, but not those appearing inside connections). If we assume that that ϕ_1 depends only on the coordinates of the compact manifold K_n of a lowest-order (Minkowski)$_d \times K_n$ supergravity solution, as is required to preserve the d-dimensional Poincaré invariance, then the dilaton equation reduces to

$$\Box\phi_1 + \frac{c}{2}\nabla^k\nabla^\ell\left(g^{ij}X_{ikj\ell}\right) = \frac{c}{2}Y. \tag{1.9}$$

The left hand side vanishes on integration over K_n, so we deduce that

$$\int_{K_n} Y = 0\,. \tag{1.10}$$

This is the same as the result that was obtained in [3] by making the assumption that M_8 is Ricci flat. As we see here, the assumption of Ricci flatness is not needed in order to prove (1.10).

The condition (1.10) is trivially satisfied if $Y = 0$, which is the case if K allows a covariantly constant SO(8) spinor χ (with respect to the Levi-Civita connection). This is because the integrability condition for such a spinor is

$$R_{ijk\ell}\,\tilde{\Gamma}^{k\ell}\,\chi = 0, \tag{1.11}$$

which implies that the matrices $R_{ijk\ell}\tilde{\Gamma}^{k\ell}$ have a common zero eigenvalue, and hence that at least one linear combination of the components of ψ in (1.6) does not appear in the integrand; the Berezin integral is then zero. However, as emphasised in [5], the vanishing of Y does not imply that there will be no correction to the Einstein equation because these corrections depend also on the the tensor X_{ij}, which vanishes only if K admits at least three Killing spinors of the same SO(8) chirality. This condition is satisfied by all hyper-Kähler 8-metrics (in agreement with the ultra-violet finiteness of supersymmetric hyper-Kähler sigma models) but not otherwise. Thus, for most cases of interest there will still be corrections due to the non-vanishing of X_{ij}. In the CY case, it was shown in [5] that the metric ceases to be Ricci-flat once these corrections are taken into account, although it remains Kähler. This result is actually a special case of a more general result; it can be shown that if M_8 has special holonomy, then

$$\tilde{X}_{ij} = 0, \qquad g^{ij}X_{ikjl} = g_{kl}Z \tag{1.12}$$

where Z is the scalar cubic in the Riemann tensor that becomes the Euler density in dimension six. In this case (1.9) is solved by

$$\phi_1 = -\frac{c}{2}Z. \tag{1.13}$$

When this is substituted into the Einstein equation, one finds that

$$R_{ij} = c\alpha'^3 \left[\nabla_i \nabla_j Z + \nabla^k \nabla^\ell X_{ikj\ell}\right]. \tag{1.14}$$

To establish this result in full generality requires consideration of 8-manifolds of Spin(7) holonomy, but this involves additional complications that we choose to postpone to a future publication. In the next section we shall establish the result for 8-manifolds of the form $M_8 = \mathbb{R} \times K_7$ with K_7 a Ricci-flat 7-manifold of G_2 holonomy; in this case we shall find that

$$X_{ikj\ell} = \frac{1}{2}\left[c_{ikm}\, c_{j\ell n}\; Z^{mn} + (i \leftrightarrow j)\right] \qquad (i, j, k, \ell = 1, \ldots, 7) \tag{1.15}$$

where c_{ikm} is the associative 3-form on the G_2 manifold, and Z^{mn} is a symmetric tensor on K_7 that arises in the analysis of this case. We shall show that this incorporates the Calabi-Yau results as a special case.

From (1.14) we see that the string corrections must deform the special holonomy metric in CY, G_2 and Spin(7) compactifications to one that is no longer Ricci flat. An important question then arises as to whether the modified solution is still supersymmetric. To answer this question, one needs to know the modifications to the supersymmetry transformation rules of the fermions, in particular the gravitino, to order α'^3. A program to determine these modifications has been underway for some years, *e.g.* [6], but the results obtained to date are not sufficient for our purposes. However, it was shown in [7, 8] for the special case of CY compactifications that there exist candidate corrections to the supersymmetry transformation of the gravitino, such that the supersymmetry of the supergravity solution is preserved in the face of the α'^3 corrections. The modified Killing spinor equation that one finds in this way naturally involves structures that are special to CY manifolds, but it was also shown in [7, 8] that the corrected Killing spinor equation can be written in a form that makes sense for any Riemannian manifold. We shall show here that this modification of the Killing spinor equation also ensures preservation of supersymmetry for modified G_2 compactifications to order α'^3.

A spacetime of the form $(\text{Minkowski})_d \times K_n$ is a solution of the $(d+n)$-dimensional supergravity theory if the K_n metric is Ricci-flat, irrespective of whether it is compact or not. The compactness of K_n is needed only for the interpretation of such solutions as Kaluza-Klein vacua; even in this case we may wish to consider a non-compact Ricci-flat manifold that approximates a given compact K_n near a singularity, and this has the advantage that an explicit metric may then be available. If so, one can explicitly determine the corrections to the K_n metric needed to maintain supersymmetry in the face of the α'^3 corrections. This issue was addressed for some six-dimensional cohomogeneity-one CY manifolds in [9], and more recently for these and further eight-dimensional CY

manifolds in [10]. The same issue for G_2 manifolds of cohomogeneity one was addressed in [9] and we take up this issue again here.

Of course, Ricci-flat manifolds of G_2 holonomy are of most interest in the context of compactifications of M-theory to four-dimensional supergravity theories with N=1 supersymmetry [11], and it is the M-theory effective action that is relevant in that case. It is known that there is a similar R^4 correction to the action of 11-dimensional supergravity, although it is more closely related to the 1-loop contribution to the IIA effective action than to the tree-level contribution, which actually has a different form. However, this difference is irrelevant for G_2 compactifications, as are the CS terms, so that our results can also be carried over to M-theory, as we explain in the final Section.

2. Corrections to G_2 Metrics

Our main concern, at least initially, is tree-level string corrections to supergravity solutions of the form $(\text{Minkowski})_2 \times M_8$, for an 8-manifold M_8 of special holonomy, and a constant dilaton. Under these circumstances, all corrections to the supergravity equations are determined by the M_8 tensor

$$X_{ij} = \frac{\delta}{\delta g^{ij}} \int d^8x \sqrt{g}\, Y. \tag{2.1}$$

where Y is given by (1.6). From (1.8) and (1.12) we expect to find that

$$X_{ij} = \nabla^k \nabla^\ell X_{ikj\ell} \tag{2.2}$$

and our first goal will be to show that if $M_8 = \mathbb{R} \times K_7$, for a 7-manifold K_7 of G_2 holonomy, then $\tilde{X}_{ij} = 0$ and the tensor X_{ijkl} takes the form (1.15).

To this end, we first write the real SO(8) Dirac matrices as $\tilde{\Gamma}^i = \tilde{\Gamma}^{\underline{i}} e_{\underline{i}}{}^i$, where $\tilde{\Gamma}^{\underline{i}}$ are the constant Dirac matrices in a frame defined by the achtbein $e_{\underline{i}}{}^i$. As there is a naturally defined '8' direction, by hypothesis, we may suppose that the achtbein is block diagonal with its 1×1 block equal to unity. We may now set

$$\tilde{\Gamma}^{\underline{i}} = \sigma_2 \otimes \Gamma^{\underline{i}} \qquad \underline{i} = 1, \ldots, 7; \qquad \tilde{\Gamma}^8 = -\sigma_1 \otimes \mathbb{1}_8, \tag{2.3}$$

where $\mathbb{1}_8$ is the 8×8 identity matrix, and then define the antisymmetric, and *imaginary* 8×8 SO(7) Dirac matrices Γ^i $(i = 1, \ldots, 7)$ from $\Gamma^{\underline{i}}$ using the 7×7 achtbein block. We choose the signs such that

$$i\Gamma^{\underline{1}} \cdots \Gamma^{\underline{7}} = \mathbb{1}_8. \tag{2.4}$$

Chiral SO(8) spinors are eigenspinors of

$$\tilde{\Gamma}_9 \equiv \tilde{\Gamma}^{\underline{1}} \cdots \tilde{\Gamma}^{\underline{8}} = \sigma_3 \otimes \mathbb{1}_8. \tag{2.5}$$

As this matrix is diagonal, we have

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \tag{2.6}$$

where $\psi_\pm$ are real 8-component SO(7) spinors such that ψ is chiral if $\psi_- = 0$ and anti-chiral if $\psi_+ = 0$.

The expression (1.6) may now be written as

$$Y \propto \int d^8\psi_+ d^8\psi_- \ \exp\left[\left(\bar{\psi}_+ \Gamma_+^{ij} \psi_+\right)\left(\bar{\psi}_- \Gamma_-^{k\ell} \psi_-\right) R_{ijk\ell}\right] \tag{2.7}$$

where from (2.3) we have the 8×8 Gamma matrices

$$\Gamma_+^{ij} = \Gamma_-^{ij} = \Gamma^{ij} = \tfrac{1}{2}\left[\Gamma^i, \Gamma^j\right], \qquad \Gamma_+^{i8} = -\Gamma_-^{i8} = i\Gamma^i \qquad (i, j = 1, \ldots, 7)\ . \tag{2.8}$$

This $\pm$ symmetric form is actually the form of this integral originally given in [3]. For the Dirac matrices as specified above, $\bar{\psi}_\pm = \psi_\pm^T$. Performing the Berezin integration one has

$$Y \propto \epsilon^{\alpha_1 \cdots \alpha_8}\, \epsilon^{\beta_1 \cdots \beta_8} \left[(\Gamma^{i_1 i_2})_{\alpha_1\alpha_2} \cdots (\Gamma^{i_7 i_8})_{\alpha_7\alpha_8}\right]\left[(\Gamma^{j_1 j_2})_{\beta_1\beta_2} \cdots (\Gamma^{j_7 j_8})_{\beta_7\beta_8}\right] \times \\ R_{i_1 i_2 j_1 j_2}\, R_{i_3 i_4 j_3 j_4}\, R_{i_5 i_6 j_5 j_6}\, R_{i_7 i_8 j_7 j_8}\,, \tag{2.9}$$

where $\alpha_1, \beta_1, \ldots$ are 8-component spinor indices of $SO(7)$.

We now take the eight-dimensional transverse space to be of the form $\mathbb{R} \times K_7$ where K_7 is a seven-manifold of G_2 holonomy. On K_7 the decomposition $G_2 \subset SO(7)$ implies that the 8-dimensional spinor representations decompose as

$$8_\pm \longrightarrow 7 + 1\,. \tag{2.10}$$

The singlet corresponds to a covariantly constant real SO(7) spinor η. It is convenient to normalise this (commuting) spinor such that $\bar{\eta}\eta = 1$, where $\bar{\eta} = \eta^T$ for a pure-imaginary representation of the $SO(7)$ Dirac matrices. A useful Fierz identity is

$$\Gamma_i\, \eta\, \bar{\eta}\, \Gamma_i + \eta\, \bar{\eta} = \mathbb{1}\,. \tag{2.11}$$

In computing the variation of Y we can make use of the zeroth-order conditions implied by the G_2 holonomy of the unperturbed background, *after* having performed the variation, of course. Let's first see why $\tilde{X}_{ij} = 0$ in (1.8) for such manifolds. The tensor $\tilde{X}_{ij}$ arises from variation of the achtbeins used to define the Dirac matrices $\tilde{\Gamma}^i$ from the constant matrices $\tilde{\Gamma}^{\underline{i}}$ (because the variation of the $\sqrt{-g}$ factor yields a

contribution proportional to Y that vanishes in the undeformed background). This leads to a term of the form

$$\delta e_{\underline{i}}{}^{i}\left[\left(\bar{\psi}_{+}\Gamma^{\underline{i}j}\psi_{+}\right)T^{+}_{ij}+\left(\bar{\psi}_{-}\Gamma^{\underline{i}j}\psi_{-}\right)T^{-}_{ij}\right] \tag{2.12}$$

in the integrand of (2.7), where $T^{\pm}_{ij}$ are tensors containing the remaining 14 independent spinors with the property that they vanish whenever any of these 14 spinors is a Killing spinor. We have two Killing spinors of opposite chirality, so in each of the above two terms, one of them must appear in the associated T-tensor, and hence the coefficient of the achtbein variation vanishes.

In contrast, variation of the metric appearing in a curvature tensor produces a non-vanishing result since the variation now involves a variation of a T-tensor, and the varied T-tensor does not have the crucial property of vanishing whenever one of its 14 spinors is a Killing spinor. Alternatively, this follows from the observation that a varied curvature need not have special holonomy (*i.e.* need not have zero eigenspinors on either front or back indices). Using the Palatini identity for the variation of the curvature, one finds that

$$\begin{aligned}\delta Y \propto \epsilon^{\alpha_1\cdots\alpha_8}\,\epsilon^{\beta_1\cdots\beta_8}\,&\left[(\Gamma^{i_1i_2})_{\alpha_1\alpha_2}\cdots(\Gamma^{i_7i_8})_{\alpha_7\alpha_8}\right]\left[(\Gamma^{j_1j_2})_{\beta_1\beta_2}\cdots(\Gamma^{j_7j_8})_{\beta_7\beta_8}\right]\times\\ &R_{i_1i_2j_1j_2}\,R_{i_3i_4j_3j_4}\,R_{i_5i_6j_5j_6}\,\nabla_{i_7}\nabla_{j_7}\,\delta g_{i_8j_8}\,.\end{aligned} \tag{2.13}$$

where the three unvaried Riemann tensors are those of the unperturbed background.

An arbitrary Majorana spinor on K_7 decomposes according to (2.10), and from (2.11) we see that $i\Gamma_i\eta$ provides a mapping between the reduced 7-component index of a spinor that is orthogonal to η, and the 7-component vector index. The SO(7) Clifford algebra is spanned by the real symmetric matrices $(\mathbb{1}, i\Gamma_{ijk})$ and the real antisymmetric matrices $(i\Gamma_i,\Gamma_{ij})$. It follows that the only independent covariantly-constant tensor that we can form from the Killing spinor spinor η is

$$c_{ijk}=\mathrm{i}\,\bar{\eta}\,\Gamma_{ijk}\,\eta\,. \tag{2.14}$$

This is the calibrating 3-form in the G_2 manifold K_7. Its Hodge dual

$$c^{ijk\ell}\equiv\tfrac{1}{6}\epsilon^{ijk\ell mnp}\,c_{mnp}=\bar{\eta}\,\Gamma_{ijk\ell}\,\eta \tag{2.15}$$

is also covariantly constant; we assume here that tensors have components with respect to an orthonormal frame. Using (2.14), (2.15), and (2.11), one can deduce that

$$c_{ijm}\,c^{k\ell m}=-c_{ij}{}^{k\ell}+2\delta^{k\ell}_{ij}\,,\qquad c_{ijmn}\,c^{k\ell mn}=-2c_{ij}{}^{k\ell}+8\delta^{k\ell}_{ij}\,, \tag{2.16}$$

The expression (2.13) for δY can now be simplified using properties of G_2 manifolds. We can choose a spinor frame in which the 8-component spinor indices decompose as

$\alpha = (\bar{\alpha}, 8)$, etc, where the covariantly-constant spinor η^α has a non-vanishing component only when $\alpha = 8$. The mapping between a 7-vector and the reduced spinor $\eta^{\bar{\alpha}}$ is then implemented with $(\Gamma_i \eta)_{\bar{\alpha}}$. Using this mapping, and making use of the G_2-manifold identity

$$R_{ijk\ell}\, c^{k\ell}{}_{mn} = 2R_{ijmn}, \tag{2.17}$$

we see that δY can be written as

$$\delta Y \propto \epsilon^{m i_1 \cdots i_6}\, \epsilon^{n j_1 \cdots j_6}\, R_{i_1 i_2 j_1 j_2}\; R_{i_3 i_4 j_3 j_4}\; R_{i_5 i_6 j_5 j_6}\, c^{ij}{}_m\, c^{k\ell}{}_n\; \nabla_i \nabla_k\, \delta g_{j\ell}. \tag{2.18}$$

The constant of proportionality can be absorbed into the constant c in (1.2), so an integration by parts in (2.1) yields the result

$$X_{ij} = c_{ikm}\, c_{j\ell n}\, \nabla^k \nabla^\ell\, Z^{mn}\,, \tag{2.19}$$

where

$$Z^{mn} \equiv \tfrac{1}{32} \epsilon^{m i_1 \cdots i_6}\, \epsilon^{n j_1 \cdots j_6}\, R_{i_1 i_2 j_1 j_2} \cdots R_{i_5 i_6 j_5 j_6}\,. \tag{2.20}$$

Note that $Z^{ij} = Z^{ji}$, and that $\nabla_i\, Z^{ij} = 0$ as a consequence of the Bianchi identity for the Riemann tensor. Note also that

$$g^{ij}\, X_{ij} = \Box Z \tag{2.21}$$

where

$$Z = g_{mn} Z^{mn}\,. \tag{2.22}$$

We thus obtain the modified Einstein equation

$$R_{ij} = c\alpha'^3 \left[\nabla_i \nabla_j Z + c_{ikm} c_{j\ell n} \nabla^k \nabla^\ell Z^{mn} \right] \tag{2.23}$$

This takes the general form (1.14) with the tensor $X_{ikj\ell}$ as given in (1.15). The dilaton is given by

$$\phi = -\tfrac{1}{2} c\, \alpha'^3\, Z\,. \tag{2.24}$$

If one takes $K_7 = \mathbb{R} \times K_6$ with K_6 a Calabi-Yau 6-manifold then the only non-zero components of the tensor Z^{ij} are $Z^{77} = Z$. As

$$c_{ij7} = J_{ij} \qquad (i, j = 1, \ldots, 6) \tag{2.25}$$

where J_{ij} is the Kähler 2-form, we have

$$R_{ij} = c\alpha'^3 \left[\nabla_i \nabla_j + J_{ik} J_{j\ell} \nabla^k \nabla^\ell \right] Z\,. \tag{2.26}$$

As Z is the Euler density for K_6, we recover the Calabi-Yau result of [5].

Although it was not necessary for the discussion we gave above, it is instructive to note that there exists a fully covariant Lagrangian that gives a modified Einstein equation which reduces to (2.23) upon restriction to a leading-order (Minkowski)$_3$ times G_2-holonomy background. Specifically, this is obtained by expanding out the product of eight-dimensional epsilon tensors in (1.5) in terms of Kronecker deltas, and then allowing the indices to range over all ten values. Terms of quadratic or higher order in the Ricci tensor or Ricci scalar coming from this expansion are unimportant when considering α'^3 corrections to the leading-order G_2 backgrounds, but the terms of linear order in Ricci tensor and scalar, together of course with the quartic Riemann tensor terms, do contribute to the order α'^3 equations of motion in these backgrounds. The terms linear in the Ricci tensor and Ricci scalar could be removed by field redefinitions, but since we have implicitly made a specific choice of field variables, the coefficients of these linear terms are now uniquely determined. Specifically, one finds that Y is given by

$$Y = W_0 - W_1 - W_2\,, \tag{2.27}$$

where

$$\begin{aligned}
W_0 &= 12(R_{a_1a_2b_1b_2}\, R^{a_2a_3b_2b_3}\, R_{a_3a_4b_4}{}^{b_1}\, R^{a_4a_1}{}_{b_3}{}^{b_4} - R_{a_1a_4a_3b_3}\, R^{a_2b_2a_3b_3}\, R^{a_1}{}_{b_1b_2b_4}\, R_{a_2}{}^{b_1a_4b_4})\,,\\
W_1 &= 6R^{ab}\,(4R_a{}^{cde}\, R_b{}^f{}_d{}^g\, R_{efcg} + 2R_a{}^d{}_b{}^e\, R_{echg}\, R_d{}^{chg} - R_a{}^{cde}\, R_{bcfg}\, R_{de}{}^{fg})\,,\\
W_2 &= R\,(R_{abcd}\, R^{cdef}\, R_{ef}{}^{ab} - 2R_{acbd}\, R^{cedf}\, R_e{}^a{}_f{}^b)\,.
\end{aligned} \tag{2.28}$$

Note that W_2 is nothing but R times the covariant expression for Z introduced in Eqs (2.20) and (2.22), namely

$$Z = R_{abcd}\, R^{cdef}\, R_{ef}{}^{ab} - 2R_{acbd}\, R^{cedf}\, R_e{}^a{}_f{}^b\,. \tag{2.29}$$

Using the Lagrangian corrections defined by (2.27), with the specific coefficients of the scheme-dependent (*i.e.* redefinition-dependent) terms W_1 and W_2, ensures that the resulting field equations yield (2.26) if K_7 is chosen to be of the form $\mathbb{R}\times K_6$ (or $S^1\times K_6$). This is a natural choice of scheme for this case, since it preserves the Kähler structure of the metric on K_6 (although it does, of course, lift the Ricci-flatness of the metric, implying that the holonomy enlarges from $SU(3)$ to $U(3)$). This can be seen directly from the corrected Ricci condition (2.26) by writing it in holomorphic coordinates, in which case it takes the form $R_{a\bar{b}} = c\alpha'^3 \partial_a \partial_{\bar{b}} Z$. The same scheme choice (2.27) also ensures that, starting from a general G_2 holonomy manifold K_7, the modified field equation is expressible as (2.23). Although in this case, unlike the Kähler case described above, there is no intermediate enlargement of the holonomy group between G_2 and the full $SO(7)$, it nevertheless seems the natural G_2 generalisation of the Kähler-preserving

Ricci correction (2.26).[1] As we shall next see, this form of the G_2 correction allows, as in the Kähler case, an elegant modification of the covariant derivative appearing in the Killing spinor equation, thus permitting supersymmetry preservation in the corrected background.

3. The Corrected Killing Spinor Equation

We seek a modified covariant derivative $\hat{\nabla}_i$ such that the 'modified G_2' space K_7 admits a (Killing) spinor η satisfying $\hat{\nabla}_i \eta = 0$ to order α'^3. We expect

$$\hat{\nabla}_i = \nabla_i + c\alpha'^3 Q_i \tag{3.1}$$

where Q_i acts by (matrix) multiplication on η. Let us define

$$Q_{ij} \equiv \nabla_i\, Q_j - \nabla_j\, Q_i. \tag{3.2}$$

Then, to order α'^3, the integrability condition $[\hat{\nabla}_i, \hat{\nabla}_j]\, \eta = 0$ gives

$$\tfrac{1}{4} R_{ijk\ell}\, \Gamma^{k\ell}\, \eta + c\alpha'^3 Q_{ij}\, \eta = 0\ . \tag{3.3}$$

Multiplying by (2.11), we see that this is equivalent to

$$R_{ijk\ell}\, c^{k\ell}{}_m + 4c\alpha'^3 \mathrm{i}\, \bar{\eta}\, \Gamma_m\, Q_{ij}\, \eta = 0\,, \qquad \bar{\eta}\, Q_{ij}\;\eta = 0\,. \tag{3.4}$$

Multiplying instead by Γ_i and then using (3.4), we also deduce that

$$R_{ij} = 2c\alpha'^3\, \bar{\eta}\, \Gamma_{jk}\, Q_i{}^k\, \eta\,. \tag{3.5}$$

Our criterion for determining Q_i will therefore be that it should satisfy $\bar{\eta}\, Q_{ij}\, \eta = 0$ and that when substituted into (3.5), it should yield the corrected field equation (2.23).

We find that the following Q_i fulfils these criteria:

$$Q_i = -\tfrac{\mathrm{i}}{2}\, c_{ijk}\, \nabla^j\, Z^{k\ell}\, \Gamma_\ell\,. \tag{3.6}$$

The verification of $\bar{\eta}\, Q_{ij}\, \eta = 0$ is immediate, and from (3.5) we find

$$\begin{aligned} R_{ij} &= -c\alpha'^3 \left(c_{jkn}\, c^{k\ell}{}_m\, \nabla_i \nabla_\ell\, Z^{mn} - c_{i\ell m}\, c_{jkn}\, \nabla^\ell \nabla^k\, Z^{mn}\right) \\ &= c\alpha'^3\, \left(\nabla_i \nabla_j\, Z + c_{i\ell m}\, c_{jkn}\, \nabla^\ell \nabla^k\, Z^{mn}\right)\ . \end{aligned} \tag{3.7}$$

[1] In particular, as we have just seen, it has the feature that if one chooses the G_2 manifold to be of the form $\mathbb{R} \times K_6$, it preserves the Kähler structure of K_6.

We have now found a modified Killing spinor equation such that a space admitting a Killing spinor must solve the modified Einstein equation to order α'^3. One would expect this condition to be equivalent, to the vanishing of the supersymmetry variation of the gravitino to order α'^3, but the latter makes sense for any background metric whereas our expression for Q_i involves the associative 3-form c_{ijk} that exists only on manifolds of G_2 holonomy. However, it is possible to rewrite Q_i in purely Riemannian terms. To do this, we first use (2.22) and the identity

$$\Gamma_i = \tfrac{\mathrm{i}}{6!}\,\epsilon_{ij_1\cdots j_6}\,\Gamma^{j_1\cdots j_6} \tag{3.8}$$

to rewrite (3.6) as

$$Q_i = \tfrac{1}{64}\,c_{ijk}\,\epsilon^{ki_1\cdots i_6}\,\nabla^j\,(R_{i_1i_2j_1j_2}\cdots R_{i_5i_6j_5j_6})\,\Gamma^{j_1\cdots j_6}\,. \tag{3.9}$$

Next, we note that

$$\begin{aligned} c_{ijk}\,\epsilon^{ki_1\cdots i_6} &= \mathrm{i}\,\bar\eta\,\Gamma_{ij}\,\Gamma_k\,\eta\,\epsilon^{ki_1\cdots i_6}\,,\\ &= -\bar\eta\,\Gamma_{ij}\,\Gamma^{i_1\cdots i_6}\,\eta\,,\\ &= 2\delta^{[i_1i_2}_{ij}\,\bar\eta\,\Gamma^{i_3\cdots i_6]}\,\eta\,,\\ &= 2\delta^{[i_1i_2}_{ij}\,c^{i_3\cdots i_6]}\,. \end{aligned} \tag{3.10}$$

From the property (2.17) we then find that (3.9) reduces to

$$Q_i = \tfrac{3}{16}\,\nabla^j\,(R_{ijm_1m_2}\,R^{k\ell}{}_{m_3m_4}\,R_{k\ell m_5m_6} - 4R_{ikm_1m_2}\,R_{j\ell m_3m_4}\,R^{k\ell}{}_{m_5m_6})\,\Gamma^{m_1\cdots m_6} \tag{3.11}$$

which is indeed purely Riemannian.

After further manipulations, which involve distributing the derivative in (3.11) and using the Bianchi identity, one can further show that Q_i can also be expressed as

$$Q_i = -\tfrac{3}{4}\,(\nabla^j\,R_{ikm_1m_2})\,R_{j\ell m_3m_4}\,R^{k\ell}{}_{m_5m_6}\,\Gamma^{m_1\cdots m_6}\,. \tag{3.12}$$

In this form, Q_i can be recognised as precisely the same modification to the Killing spinor condition that was proposed in [7]. In that case, the proposal was based on the consideration of deformations from $SU(3)$ holonomy for six-dimensional Calabi-Yau backgrounds, so there was no a priori reason to expect the same expression in the G_2 case.

4. Explicit Examples

In this Section, we shall consider some examples of 7-dimensional metrics of cohomogeneity-one, and, by making use of the modified Killing spinor equation

$$\left(\nabla_i - \frac{i}{2}\alpha'^3 c_{ijk}\nabla^j Z^{k\ell}\,\Gamma_\ell\right)\eta = 0\ . \tag{4.1}$$

Note that we set $c = 1$ in this Section, as this can always be arranged by a choice of units for α'. We use this Killing spinor condition to derive first-order equations for the metric coefficients such that the metrics will give supersymmetric solutions of the modified Einstein equations (2.23).[2]

4.1 $S^3 \times S^3$ principal orbits with $SU(2)^3$ symmetry

For our first example, we take the case of cohomogeneity-one metrics with $S^3 \times S^3$ principal orbits and $SU(2)^3$ isometry. This class of metrics was considered in [14, 15], where it was shown that there exists a complete, non-singular Ricci-flat solution with G_2 holonomy. One starts from the metric ansatz

$$ds^2 = dt^2 + a^2\,(\sigma_i - \Sigma_i)^2 + b^2\,(\sigma_i + \Sigma_i)^2\,, \tag{4.2}$$

where σ_i and Σ_i are two independent sets of left-invariant 1-forms for the group $SU(2)$, and a and b are functions of t. Since the undeformed G_2 metric, and hence also the deformed one, admits a single Killing spinor η, it follows that in the natural basis for the vielbein and the spin frame, this spinor must be independent of the coordinates on the $S^3 \times S^3$ orbits. We shall take

$$e^0 = dt\,, \qquad e^i = a\,(\sigma_i - \Sigma_i)\,, \qquad e^{\hat{i}} = b\,(\sigma_i + \Sigma_i)\,, \tag{4.3}$$

from which it follows that

$$\begin{aligned}
&\omega_{0i} = -\frac{\dot{a}}{a}\,e^i\,, \qquad \omega_{0\hat{i}} = -\frac{\dot{b}}{b}\,e^{\hat{i}}\,, \qquad \omega_{ij} = (\frac{b}{4a^2} - \frac{1}{2b})\,\epsilon_{ijk}\,e^{\hat{k}}\,,\\
&\omega_{\hat{i}\hat{j}} = -\frac{1}{4b}\,\epsilon_{ijk}\,e^k\,, \qquad \omega_{i\hat{j}} = -\frac{b}{4a^2}\,\epsilon_{ijk}\,e^k\,.
\end{aligned} \tag{4.4}$$

After calculating D_i in this frame one sees easily that η will be constant.

It now becomes a straightforward matter to read off the equations that follow from requiring that such a constant η satisfy the Killing spinor equation (4.1). As usual, we may substitute the uncorrected first-order equations for G_2 holonomy when evaluating the correction term Q_i in (3.1), given in (3.6), since we are working here only to order α'^3, and there is already an explicit factor of α'^3 associated with this term. Thus the term Q_i can be expressed in purely algebraic terms. We find the following first-order conditions:

$$\frac{\dot{a}}{a} + \frac{b}{2a^2} + \dot{S}_1 = 0\,, \qquad \frac{\dot{b}}{b} - \frac{b}{4a^2} + \frac{1}{4b} + \dot{S}_2 = 0\,, \tag{4.5}$$

[2]Details of the uncorrected metrics and expressions for the tensors c_{ijk} can be found in Refs [14, 15, 16, 17].

where

$$
\begin{aligned}
S_1 &= -\alpha'^3 \tfrac{1}{9216}\, a^{-12}\, (6851 a^6 - 38274 a^4\, b^2 + 69147 a^2\, b^4 - 39312 b^6)\,,\\
S_2 &= \alpha'^3 \tfrac{5}{9216}\, a^{-12}\, (463 a^6 - 2892 a^4\, b^2 + 4563 a^2\, b^4 - 1890 b^6)\,.
\end{aligned}
\tag{4.6}
$$

Note that the α'^3 terms are total time derivatives. In their absence, equations (4.5) reduce precisely to the standard first-order conditions for G_2 holonomy, yielding the non-singular solution obtained in [14, 15]. We want to determine how this solution is modified by the α'^3 terms. Defining a new radial variable ρ by $dt = -b^{-1}\, d\rho$, and letting $A \equiv a^2$, $B \equiv b^2$, the equations (4.5) become

$$
\frac{dA}{d\rho} + 2A\, \frac{dS_1}{d\rho} = 1\,, \qquad \frac{1}{B}\, \frac{dB}{d\rho} - \frac{1}{2B} + \frac{df}{d\rho} = 0\,, \tag{4.7}
$$

where

$$
f = \tfrac{1}{2} \log A + S_1 + 2 S_2\,. \tag{4.8}
$$

After some simple manipulations we therefore arrive at the solution

$$
\begin{aligned}
a^2(\rho) &= e^{-2S_1(\rho)}\, \int_{\rho_1}^{\rho} e^{2S_1(x)}\, dx\,,\\
b^2(\rho) &= \frac{1}{2a(\rho)}\, e^{-[S_1(\rho) + 2S_2(\rho)]}\, \int_{\rho_0}^{\rho} a(x)\, e^{[S_1(x) + 2S_2(x)]}\, dx\,.
\end{aligned}
\tag{4.9}
$$

Regular solutions arise if we take $\rho_1 < \rho_0$, so that as ρ runs from the asymptotic region $\rho = \infty$ downwards, the metric function b vanishes while a is still non-zero. Provided that $\rho_1 < \rho_0$, the precise choice of value for ρ_1 is arbitrary, since the system of equations has a shift symmetry $\rho \longrightarrow \rho + \text{constant}$. A convenient choice is to take ρ_1 to be the constant such that $a^2(\rho)$ in (4.9) is given by

$$
a^2(\rho) = e^{-2S_1(\rho)}\, \left[\rho + \int_{\rho}^{\infty} \left(1 - e^{2S_1(x)}\right)\, dx\right]\,. \tag{4.10}
$$

The solution (4.9) to (4.5) would be exact if we viewed S_1 and S_2 as arbitrarily-specified external "source functions." In fact, of course, S_1 and S_2 are themselves given by (4.6), and so (4.9) should be viewed as integro-differential equations governing the functions a and b.

The equations (4.9) can easily be solved up to the order α'^3 to which we are working, since one can simply substitute the leading-order solutions for the undeformed G_2 holonomy metric into the expressions (4.6) for S_1 and S_2, since they already carry an explicit factor of α'^3. These leading-order solutions can themselves be read off from (4.9) by setting S_1 and S_2 to zero, yielding

$$
a^2 = \rho\,, \qquad b^2 = \tfrac{1}{3}\rho\, [1 - (\rho_0/\rho)^{3/2}]\,. \tag{4.11}
$$

These expressions give the standard complete, non-singular metric of G_2 holonomy found in [14, 15]. The explicit solution with α'^3 correction can then be obtained straightforwardly using (4.9), where S_1 and S_2 take their Ricci-flat background forms. It is given by

$$\begin{aligned} a^2 &= \rho\,\Bigg(1+\tfrac{1}{32}\alpha'^3\,\Bigg[\frac{10}{9\rho^3}+\frac{220\rho_0^{3/2}}{63\rho^{9/2}}+\frac{221\rho_0^3}{48\rho^6}+\frac{14\rho_0^{9/2}}{9\rho^{15/2}}\Bigg]\Bigg)\,,\\ b^2 &= \tfrac13\rho\,\big[1-(\rho_0/\rho)^{3/2}\big]\,\Bigg(1+\tfrac{1}{32}\alpha'^3\Bigg[-\frac{4589}{2016(\rho\,\rho_0)^{3/2}}+\frac{6611}{2016\rho^3}-\frac{863\rho_0^{3/2}}{672\rho^{9/2}}\\ &\qquad -\frac{451\rho_0^3}{288\rho^6}+\frac{133\rho_0^{9/2}}{72\rho^{15/2}}\Bigg]\Bigg)\,. \end{aligned} \tag{4.12}$$

This solution is finite everywhere from $\rho=\rho_0$ to $\rho=\infty$.

It should be noted that the solution (4.12) does not have a smooth $\rho_0\to 0$ limit. In fact, it is of interest to look in closer detail at what happens if the parameter ρ_0 in the original G_2 holonomy solution (4.11) is taken to be zero, implying $b^2=\frac13 a^2$. This corresponds to the situation where the cohomogeneity-one metric is nothing but the cone over $S^3\times S^3$, since the G_2 metric is then

$$ds^2 = dt^2+\tfrac{1}{12}t^2\,\Big((\sigma_i-\Sigma_i)^2+\tfrac13(\sigma_i+\Sigma_i)^2\Big)\,. \tag{4.13}$$

The quantity in the large parentheses is the Einstein metric of weak $SU(3)$ holonomy on $S^3\times S^3$. The metric (4.13) is, of course, singular at $t=0$, the apex of the cone. If we now take this leading-order solution, and follow the same procedure of studying the higher-order corrections up to order α'^3, the calculations become very simple. In fact with $b^2=\frac13 a^2$, the first-order equations (4.5) reduce, after substituting the leading-order solution into the expressions for S_1 and S_2, to

$$\dot a = \frac{\sqrt3}{6}\Big(\frac{5\alpha'^3}{24a^6}-1\Big)\,. \tag{4.14}$$

This can be solved by defining a new radial variable r such that

$$dr = \Big[(5/24)\alpha'^3\,a^{-6}-1\Big]\,dt\,, \tag{4.15}$$

whereupon the metric becomes[3]

$$ds^2 = \frac{dr^2}{\left(1 - \frac{m^6}{r^6}\right)^2} + \tfrac{1}{12} r^2 \left((\sigma_i - \Sigma_i)^2 + \tfrac{1}{3} (\sigma_i + \Sigma_i)^2 \right) , \tag{4.16}$$

where $m^2 = 3(45)^{1/3}\,\alpha'$.

This metric is smooth as one descends from large r to a minimum value at $r = m$, which is at infinite affine distance. This is suggestive of what one might expect from string theory given that string theory generally permits conical singularities. Of course, the metric should really be expanded in powers of m^2, and only the leading, m^6, correction retained, since the higher-order terms in this expansion are of the same order of magnitude as others we have neglected. Actually, the expansion parameter here is m^2/r^2, on dimensional grounds, so for any finite m we would never be justified in considering $r \sim m$; the geometry in a region of size m near what was the apex of the cone in the unperturbed, supergravity, solution is inaccessible within the α' expansion. However, it is clear that string corrections do change the supergravity result in this region, and it is plausible that this effect is captured by the metric (4.16).

4.2 $S^3 \times S^3$ principal orbits with $SU(2)^2 \times U(1)$ symmetry

For the next example, consider

$$ds^2 = dt^2 + a_i^2\,(\sigma_i - \Sigma_i)^2 + b_i^2 (\sigma_i + \Sigma_i)^2 \tag{4.17}$$

where the σ_i and Σ_i are again the left-invariant 1-forms of $SU(2) \times SU(2)$, and we split the triplet index as $i = (1, \alpha)$. The first-order equations can be expressed as the following

$$\begin{aligned} \frac{a_1'}{a_1} - \frac{a_1}{4a_3\,b_1} + \frac{a_3}{4a_1\,b_2} + \frac{b_2}{4a_1\,a_3} - \frac{a_1}{4a_2\,b_3} + \frac{a_2}{4a_1\,b_3} + \frac{b_3}{4a_1\,a_2} + K_1 = 0\,, \\ \frac{b_1'}{b_1} + \frac{a_2}{4a_3\,b_1} + \frac{a_3}{4a_2\,b_1} - \frac{b_1}{4a_2\,a_3} + \frac{b_1}{4b_2\,b_3} - \frac{b_2}{4b_1\,b_3} - \frac{b_3}{4b_1\,b_2} + \widetilde{K}_1 = 0\,, \end{aligned} \tag{4.18}$$

together with the cyclic permutations of indices $1, 2$ and 3. The equations for the leading-order Ricci-flat metric were obtained in [16]. We have worked out explicitly the result for the ${\alpha'}^3$ contributions K_i and $\widetilde{K}_i$; however, they are too complicated to

[3]Results for the modification of G_2 holonomy metrics given in [9] suggested that the singular cone-metric limits of the regular cohomogeneity-one G_2 metrics would not suffer ${\alpha'}^3$ corrections. By contrast, we find non-vanishing modifications in the cone-metric limits for all the known G_2 holonomy examples, and indeed our results in general differ from those in [9]. The apparent discrepancy between our results and those in [9] may be due to a different choice of scheme.

present here since they involve thousands of terms. In [17], an analytic solution for a special case $a_3 = a_2$ and $b_3 = b_2$ was obtained; in terms of the new radial variable r defined by

$$dt = -\sqrt{\frac{9(r-\ell)(r+\ell)}{4(r-3\ell)(r+3\ell)}}\, dr \equiv h(r)dr \tag{4.19}$$

we have

$$\begin{aligned} a_1 &= -\tfrac{1}{2} r\,, \qquad a_2 = a_3 = \sqrt{\tfrac{3}{16}(r-\ell)(r+3\ell)} \\ b_1 &= \ell\sqrt{\frac{(r-3\ell)(r+3\ell)}{(r-\ell)(r+\ell)}}\,, \qquad b_2 = b_3 = -\sqrt{\tfrac{3}{16}(r+\ell)(r-3\ell)}. \end{aligned} \tag{4.20}$$

For this Ricci-flat G_2 background, we have

$$\begin{aligned} h\,K_1 &= -\frac{327680\alpha'^3\,\ell^2\,r(r^4 - r^2\,\ell^2 + 32\ell^4)}{243(r^2-\ell^2)^7}\,, \\ h\,K_2 &= \frac{81920\alpha'^3\,\ell^2\,(r^6 - 14r^2\,\ell + 35r^4\,\ell^2 - 64r^3\,\ell^3 - 77r^2\,\ell^4 - 66r\,\ell - 39\ell^6}{243(r-\ell)^8(r+\ell)^7}\,, \\ h\,\widetilde{K}_1 &= \frac{327689\alpha'^3\,\ell^2\,r(r^6 - 20r^4\,\ell^2 + 237r^2\,\ell^4 + 118\ell^6)}{243(r^2-\ell^2)^8}\,, \\ h\,\widetilde{K}_2 &= \frac{81920\alpha'^3\,\ell^2\,(r^6 + 14r^5\,\ell + 35r^4\,\ell^2 + 64r^3\,\ell^3 - 77r^2\,\ell^4 + 66r\,\ell^5 - 39\ell^6)}{243(r-\ell)^7\,(r+\ell)^8}\,, \end{aligned} \tag{4.21}$$

with $K_3 = K_2$ and $\widetilde{K}_3 = \widetilde{K}_2$. The α'^3 correction to the metric of G_2 holonomy could now be worked out explicitly as in the previous example.

4.3 $\mathbb{CP}^3$ principal orbits

The simplest description of these metrics is the one given in [16], in which the left-invariant 1-forms $L_{AB} = -L_{BA}$ of $SO(5)$, satisfying $dL_{AB} = L_{AC} \wedge L_{CB}$, are decomposed into

$$R_i = \tfrac{1}{2}(L_{0i} + \tfrac{1}{2}\epsilon_{ijk}\,L_{jk})\,, \qquad L_i = \tfrac{1}{2}(L_{0i} - \tfrac{1}{2}\epsilon_{ijk}\,L_{jk})\,, \qquad P_a = L_{a4}\,. \tag{4.22}$$

Here the index $0 \le A \le 4$ is split as $A = (a, 4) = (0, i, 4)$, and so the L_i and R_i are left-invariant 1-forms of the $SO(4) \sim SU(2)_L \times SU(2)_R$ subalgebra, and P_a are in the coset $S^4 = SO(5)/SO(4)$. Thus the 1-forms (R_i, P_a) span S^7 described as an $SU(2)$ bundle over S^4, and so (R_1, R_2, P_a) span $\mathbb{CP}^3 = S^7/U(1)$, viewed as an S^2 bundle over S^4. The ansatz for cohomogeneity-one metrics with $\mathbb{CP}^3$ principal orbits is then

$$ds^2 = dt^2 + a^2\,P_a^2 + b^2\,(R_1^2 + R_2^2)\,, \tag{4.23}$$

where a and b are functions of t.

Following the same procedure as in Section 4.1, we find that the first-order equations for a and b are

$$\frac{\dot{a}}{a} + \frac{b}{2a^2} + \dot{S}_1 = 0\,, \qquad \frac{\dot{b}}{b} - \frac{b}{2a^2} + \frac{2}{b} + \dot{S}_2 = 0\,, \tag{4.24}$$

where S_1 and S_2 are given by

$$\begin{aligned} S_1 &= -\tfrac{9}{32}\alpha'^3\, a^{-12}\,(4a^2-b^2)(256a^4-234a^2\,b^2+51b^4)\,,\\ S_2 &= \tfrac{9}{16}\alpha'^3\, a^{-12}\,(8a^2-3b^2)(44a^4-38a^2\,b^2+9b^4)\,. \end{aligned} \tag{4.25}$$

These equations can be solved by introducing a new radial variable ρ, defined by $dt = -b^{-1}\,d\rho$. After analogous manipulations to those described in Section 4.1, we find that a and b are given by

$$\begin{aligned} a^2(\rho) &= e^{-2S_1(\rho)}\int_{\rho_1}^{\rho} e^{2S_1(x)}\,dx\,,\\ b^2(\rho) &= \frac{4}{a^2(\rho)}\, e^{-2[S_1(\rho)+S_2(\rho)]}\int_{\rho_0}^{\rho} a^2(x)\, e^{2[S_1(x)+S_2(x)]}\,dx\,. \end{aligned} \tag{4.26}$$

As in Section 4.1, we should take $\rho_1 < \rho_0$, with the simplest choice being such that the expression for a^2 in (4.26) is given by (4.10). Again, these exact integro-differential equations can be solved easily, at order α'^3, since one can substitute the leading-order solutions $a^2=\rho$, $b^2 = 2\rho(1-\rho_0^2/\rho^2)$ of the G_2-holonomy metric (as found in [14, 15]) into the expressions (4.25) for S_1 and S_2. We find that

$$\begin{aligned} a^2 &= \rho + (9/16)\,\alpha'^3\,\rho^{-8}\left(-24\rho^6+130\rho^4\,\rho_0^2+616\rho^2\,\rho_0^4+459\rho_0^6\right)\\ b^2 &= 2\rho\,\left[1-\left(\rho_0^2/\rho^2\right)\right] + (9/8)\,\alpha'^3\,(\rho-\rho_0)^2\,\rho_0^{-1}\rho^{-10}\Big(326\rho^7+628\rho^6\,\rho_0+930\rho^5\,\rho_0^2\\ &\qquad +1090\rho^4\,\rho_0^3+1250\rho^3\,\rho_0^4+1108\rho^2\,\rho_0^5+966\rho\,\rho_0^6+483\rho_0^7\Big)\,. \end{aligned} \tag{4.27}$$

4.4 $SU(3)/U(1)^2$ flag-manifold principal orbits

In this example, it is convenient, as in [16] to introduce the (complex) left-invariant 1-forms ${L_A}^B$ of $SU(3)$, which satisfy $({L_A}^B)^\dagger = {L_B}^A$, ${L_A}^A=0$ and $d{L_A}^B = \mathrm{i}\,{L_A}^C\wedge {L_C}^B$, where $A=1,2,3$. The six real 1-forms defined by

$$\sigma_1+\mathrm{i}\,\sigma_2 \equiv {L_1}^3\,,\quad \Sigma_1+\mathrm{i}\,\Sigma_2\equiv {L_2}^3\,,\quad \nu_1+\mathrm{i}\,\nu_2\equiv {L_1}^2\,, \tag{4.28}$$

span the coset $SU(3)/(U(1)\times U(1))$ of the flag manifold.

4.4.1 Biaxial ansatz

First, we consider a biaxial ansatz for the metric, with

$$ds^2 = dt^2 + a^2\,(\sigma_1^2 + \sigma_2^2 + \Sigma_1^2 + \Sigma_2^2) + b^2(\nu_1^2 + \nu_2^2)\,, \tag{4.29}$$

where a and b are functions of t.

Following the same procedures as in the previous examples, we find that the first-order equations for the existence of a singlet Killing spinor are given by

$$\frac{\dot a}{a} - \frac{b}{a^2} + \dot S_1 = 0\,, \qquad \frac{\dot b}{b} + \frac{b}{a^2} - \frac{2}{b} + \dot S_2 = 0\,, \tag{4.30}$$

where S_1 and S_2 are given by

$$\begin{aligned} S_1 &= -54\alpha'^3\, a^{-12}\,(2a^2 - b^2)(20a^4 - 39a^2\, b^2 + 17b^4)\,,\\ S_2 &= 108\alpha'^3\, a^{-12}\,(20a^6 - 41a^4\, b^2 + 31a^2\, b^4 - 9b^6)\,. \end{aligned} \tag{4.31}$$

Following analogous procedures to those applied above, we define a new radial coordinate here by $dt = b^{-1}\, d\rho$, leading to the solution

$$\begin{aligned} a^2(\rho) &= 2e^{-2S_1(\rho)} \int_{\rho_1}^{\rho} e^{2S_1(x)}\, dx\,,\\ b^2(\rho) &= \frac{4}{a^2(\rho)}\, e^{-2[S_1(\rho)+S_2(\rho)]} \int_{\rho_0}^{\rho} a^2(x)\, e^{[2S_1(x)+S_2(x)]}\, dx\,. \end{aligned} \tag{4.32}$$

As in the previous examples, we can choose ρ_1 so that we here have

$$a^2(\rho) = 2e^{-2S_1(\rho)}\, \Big[\rho + \int_\rho^\infty \big(1 - e^{2S_1(x)}\big)\, dx\Big]\,. \tag{4.33}$$

In this case the solution to order α'^3 is

$$\begin{aligned} a^2 &= 2\rho + (9/8)\,\alpha'^3\,\big(-72\rho^6 + 90\rho^4\,\rho_0^2 + 616\rho^2\,\rho_0^4 - 459\rho_0^6\big)\,,\\ b^2 &= 2\rho\,\big[1 - \big(\rho_0^2/\rho^2\big)\big] + (9/8)\,\alpha'^3\,(\rho-\rho_0)^2\,\rho_0^{-1}\rho^{-10}\Big(502\rho^7 + 932\rho^6\,\rho_0 + 1362\rho^5\,\rho_0^2\\ &\qquad + 1306\rho^4\,\rho_0^3 + 1250\rho^3\,\rho_0^4 + 1108\rho^2\,\rho_0^5 + 966\rho\,\rho_0^6 + 483\rho_0^7\Big)\,. \end{aligned} \tag{4.34}$$

4.4.2 Triaxial ansatz

The biaxial Flag metric ansatz (4.29) can be generalised to a triaxial ansatz, with

$$ds^2 = dt^2 + a^2\,(\sigma_1^2 + \sigma_2^2) + b^2(\Sigma_1^2 + \Sigma_2^2) + c^2(\nu_1^2 + \nu_2^2)\,, \tag{4.35}$$

We find that the first-order equations are

$$\frac{\dot{a}}{a} = \frac{a^2 - b^2 - c^2}{a\, b\, c} - \dot{S}_1 \tag{4.36}$$

and its cyclic permutations with S_1, S_2, S_3, where

$$\begin{aligned} S_1 = & -\frac{54(a^2+b^2-c^2)(a^2-b^2+c^2)}{a^6\, b^6\, c^6}\Big[18a^8 + 17b^8 + 17c^8 \\ & -31a^6\, b^2 - 29a^2\, b^6 - 31a^6\, c^2 - 10b^6\, c^2 - 29a^2\, c^6 - 10b^2 c^6 + 25a^4\, b^4 \\ & +25a^4\, c^4 - 14b^4\, c^4 + 32a^4\, b^2\, c^2 + 9a^2\, b^4\, c^2 + 25a^2\, b^2\, c^4\Big] \end{aligned} \tag{4.37}$$

and cyclic permutations for S_2 and S_3. These equations may be simplified by defining

$$u_1 = b\, c\,, \qquad u_2 = c\, a\,, \qquad u_3 = a\, b\,, \tag{4.38}$$

and introducing the new radial coordinate ρ such that $d\rho = -abc\, dt$. The equations then become

$$u_1' - \frac{2}{u_1} = -u_1\ (S_2' + S_3') \tag{4.39}$$

and cyclic permutations, where the prime indicates differentiation with respect to ρ. These equations can be integrated to give

$$u_1^2(\rho) = 4e^{-2S_2(\rho)-2S_3(\rho)} \int_{\rho_1}^{\rho} e^{2S_2(x)+2S_3(x)}\, dx \tag{4.40}$$

and cyclic permutations. The functions $u_!, u_2, u_3$, and hence the metric coefficients a, b, c may then be found to order α'^3 as before.

5. Lifting to M-theory

So far we have focused on the α'^3 corrections at tree level in type II string theory, which are identical for type IIA and type IIB. The IIA string can be viewed as an S^1 compactification of M-theory, and to this extent our results are also of direct relevance in M-theory. However, all tree-level α' corrections to the string effective action vanish in the decompactification limit in which we recover uncompactified M-theory. Nevertheless, there *are* R^4 corrections in M-theory, which arise in the context of IIA string theory as *one-loop* corrections to the $\alpha'^3 R^4$ term in the effective action. The IIB contribution to the R^4 term at one loop has the same structure as the tree-level term, as required by $Sl(2;Z)$ duality, but the IIB and IIA contributions are no longer identical at one-loop because of the different Ramond-Ramond sectors circulating in the loop.

The situation can be summarised as follows (see, for example, [12, 13]). From (1.5) one sees that Y can be written as

$$Y = Y_0 - E_8 \tag{5.1}$$

where $Y_0 \sim t_8 t_8 R^4$ and E_8 is the 10-dimensional scalar that reduces to the Euler density on M_8. The α'^3 contributions to the type IIA and type IIB effective Lagrangians at tree-level and 1-loop can now be summarised by the following table, where CS stands for a $B \wedge t_8 R^4$ Chern-Simons term [18]:

Table 1: String tree-level and one-loop corrections

	Tree Level	One Loop
IIA:	$e^{-2\phi}\,(Y_0 - E_8)$	$Y_0 + E_8 + \text{CS}$
IIB:	$e^{-2\phi}\,(Y_0 - E_8)$	$Y_0 - E_8$

The one-loop CS contribution to the IIA theory lifts to a similar $A \wedge t_8 R^4$ term in eleven dimensions [19]. Fortunately, this term is irrelevant to G_2 compactifications, as is the E_8 term, in view of their $n = 7$ dimensionality. For our present purposes, we may consistently set the 4-form field strength to zero, in which case the relevant part of the (bosonic) M-theory effective Lagrangian density can therefore be taken to be just

$$\mathcal{L}_{11} = \sqrt{-\hat{g}}\,\Big[\hat{R} - \lambda\,\hat{Y}_0\Big]\,, \tag{5.2}$$

where $\lambda \sim \ell_{11}^6$ must be small in comparison to the length scale of the compactifying solution. It should be understood that the hatted quantities are eleven dimensional. Of course, this also implies a particular choice of field variables; one could also include terms that vanish by use of the leading-order field equation $\hat{R}_{MN} = 0$. Clearly, the choice of whether to include such terms or not is a matter of taste, related to the desired form of the resulting field equations. Indeed, in the following we shall make a specific choice of field redefinition that has the virtue of producing the same form of the internal space deformations as we obtained previously in Section 2 at string tree level. We recall that, in particular, this has the feature that for an internal space of the form $K_7 = S^1 \times K_6$, the initially Ricci-flat Kähler space K_6 remains Kähler even after the quantum deformation.[4]

In order to apply our previous results directly in eleven dimensions, we now consider this field redefinition in detail. The need for this redefinition arises from the fact that in our string theory discussion of Section 2, the eventual form (2.23) for the modification

[4]This scheme choice can be contrasted with that made in Ref. [9], which does not preserve the Kähler condition.

of the Ricci-flatness condition on the G_2 manifold included a contribution from an α'^3 correction to the dilaton. In eleven dimensions, where there is no dilaton, the equations of motion that follow from (5.2) would therefore imply a different equation from (2.23) for the corrected G_2 metric. We can compensate for the absence of the dilaton by adding a term proportional to $\hat{R}\,\hat{Z}$, where $\hat{Z}$ is given by (2.29) evaluated in eleven dimensions, achieved by performing the field redefinition

$$\hat{g}_{MN} \longrightarrow (1 + \tfrac{2}{9}\lambda\,\hat{R}\,\hat{Z})\,\hat{g}_{MN}\,. \tag{5.3}$$

After this field redefinition, the Lagrangian density (5.2) becomes

$$\begin{aligned}\mathcal{L}_{11} &= \sqrt{-\hat{g}}\,\Big[\hat{R} - \lambda\,\hat{Y}_0 + \lambda\,\hat{R}\,\hat{Z}\Big]\,,\\ &= \sqrt{-\hat{g}}\,\Big[\hat{R} - \lambda\,(\hat{W}_0 - \hat{W}_1 - 2\hat{W}_2)\Big]\,.\end{aligned} \tag{5.4}$$

It easily verified that the resulting field equations are solved by backgrounds of the form (Minkowski)$_4 \times K_7$, where K_7 has a modified G_2 metric satisfying the same equation (2.23) as we had for the (Minkowski)$_3 \times K_7$ metrics in string theory. Note that since λ is itself "small," and $\hat{R}$ is of order λ, when varying the additional term in (5.4) we need only retain the terms coming from $\delta\hat{R} = (-\hat{\nabla}_M\,\hat{\nabla}_N + \hat{g}_{MN}\,\hat{\Box})\,\delta\hat{g}^{MN} + \cdots$.

The modified $D=11$ equations following from (5.4) with leading-order correction terms specialised to the (Minkowski)$_4 \times K_7$ case take the form

$$\hat{R}_{\mu\nu} - \tfrac{1}{2}\hat{R}\hat{g}_{\mu\nu} = -\lambda g_{\mu\nu}\Box Z \tag{5.5}$$

$$\hat{R}_{ij} - \tfrac{1}{2}\hat{R}\hat{g}_{ij} = \lambda(X_{ij} + \nabla_i\nabla_j Z - g_{ij}\Box Z)\,, \tag{5.6}$$

where X_{ij} was given in Eq. (2.19). For $D=11$ spacetimes of the form (Minkowski)$_4 \times K_7$, we have $\hat{R} = R$, where R is the 7-dimensional Ricci scalar. Using (5.6) and (2.21), one has $R = 2\lambda\Box Z$ and hence

$$R_{ij} = \lambda(X_{ij} + \nabla_i\nabla_j Z)\,. \tag{5.7}$$

Eq. (5.5) is then identically satisfied for a (Minkowski)$_4 \times K_7$ spacetime, because it simply reduces to $-\lambda\Box Z\eta_{\mu\nu} = -\lambda\Box Z\eta_{\mu\nu}$. Accordingly, the M-theory deformation of a G_2 internal manifold as given in Eq. (5.7) is exactly the same as that given by (2.23) for the $D=10$ string tree-level case.

If one uses the standard Kaluza-Klein ansatz for reduction to $D=10$

$$d\hat{s}_{11}^2 = e^{-\frac{2}{3}\tilde{\phi}}ds_{10}^2 + e^{\frac{4}{3}\tilde{\phi}}\,dy^2\,, \tag{5.8}$$

then one finds the Lagrangian density

$$\mathcal{L}_{10} = \sqrt{-g}\left\{e^{-2\tilde{\phi}}\left[R + 4\left(\partial\tilde{\phi}\right)^2\right] - \lambda\,Y_0 + \lambda\,Z\,(R + 4\Box\tilde{\phi})\right\} + \cdots\,, \tag{5.9}$$

where the ellipses represent terms of higher order in the small parameter λ, which are unimportant at the order to which we are working. The trickiest part of this calculation is the evaluation of the term $\Box\tilde{\phi}\, Z$, which arises from the Kaluza-Klein reduction of $\hat{Y}_0$ and $\hat{W}_2 = \hat{R}\,\hat{Z}$. It is useful to note that at the linearised level, which suffices for our purposes, the reduction of the Riemann tensor gives

$$\hat{R}_{\mu\nu\rho\sigma} = e^{\frac{2}{3}\tilde{\phi}}\, R_{\mu\nu\rho\sigma} - \tfrac{1}{3}(\nabla_\rho\nabla_\nu\tilde{\phi}\, g_{\mu\sigma} - \nabla_\sigma\nabla_\nu\tilde{\phi}\, g_{\mu\rho} - \nabla_\rho\nabla_\mu\tilde{\phi}\, g_{\nu\sigma} + \nabla_\sigma\nabla_\mu\tilde{\phi}\, g_{\nu\rho})\,, \quad (5.10)$$

and that the substitution of this into $\hat{Y}_0$ can be evaluated using the variational formula (2.1) and the relation (2.21) which holds in the original G_2 background.

The terms involving Z in (5.9) can then be removed by redefining the dilaton according to

$$\tilde{\phi} = \phi + \tfrac{1}{2}\lambda\, Z\,, \quad (5.11)$$

which implies that (5.9) becomes

$$\mathcal{L}_{10} = \sqrt{-g}\,\left\{e^{-2\phi}\left[R + 4\,(\partial\phi)^2\right] - \lambda\, Y_0\right\} + \cdots\,, \quad (5.12)$$

This is the part of the α'^3 corrected effective action given in Table 1 at one string loop for the type IIA theory that is relevant to G_2 compactifications.[5] The dilaton redefinition (5.11) is easily understandable when one recalls that the field $\tilde{\phi}$ appearing in the Kaluza-Klein reduction ansatz (5.8) is constant in the (Minkowski)$_4 \times K_7$ background, whereas the redefined dilaton ϕ appearing in (5.11) is non-constant, and precisely reproduces what we found in (1.13) in the string compactification discussion of Section 2.

If one tried to oxidise directly the $D = 10$ corrected Lagrangian $\mathcal{L}_{10}$ of Eq. (5.12) back to $D = 11$ via the standard Kaluza-Klein ansatz $d\hat{s}_{11}^2 = e^{-\frac{2}{3}\phi} ds_{10}^2 + e^{\frac{4}{3}\phi}\, dy^2$, one would not obtain a manifestly $D = 11$ covariant result, since $g_{11\,11}$ would appear asymmetrically in the correction terms. Moreover, the (Minkowski)$_3 \times K_7$ vacuum of the $D = 10$ theory would not lift via this oxidation up to a manifestly $D = 4$ Lorentz invariant (Minkowski)$_4 \times K_7$ solution. The field redefinition (5.11) resolves these issues, allowing one to pass between the quantum corrected $D = 11$ covariant action (5.4) and the corrected $D = 10$ string action (5.12), and similarly for the (Minkowski)$\times K_7$ solutions. Of course, one could equivalently wrap the field redefinition (5.8) into the Kaluza-Klein ansatz and so view the passage between the $D = 11$ and $D = 10$ theories to be via a quantum corrected KK ansatz.

We have thus shown (in close analogy to string theory) that the R^4 corrections to M-theory preserve the supersymmetry of the Kaluza-Klein vacua of 11-dimensional supergravity of the form (Minkowski)$_4 \times K_7$ in which K_7 is a (Ricci flat) 7-manifold of

[5]The tree-level contribution arises from consideration of the Kaluza-Klein modes [20].

G2 holonomy, although (as in $D = 10$ string theory) these corrections do not in fact preserve either the Ricci flatness of K_7 or its G_2 holonomy. This circumstance may suggest that an appropriate perspective for interpreting the preservation of unbroken supersymmetry might be that of generalised holonomies [21].

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